Convergence of Series

Students should be able to:

- Recognize various types of numerical series and efficiently apply the appropriate test.
- Determine the sum of an infinite geometric series and be able to use that sum to create a power series and determine its interval of convergence.
- Use substitution, differentiation, and integration to create series related to given series.
- Use the Ratio Test to determine radius or open interval of convergence of power series.
- Use the other tests to check convergence at the endpoints.

Test	Series	Converges	Diverges	Notes
n^{th} term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	
Geometric Series	$\sum_{n=1}^{\infty} a \cdot r^{n-1}$	$-1 < r < 1$	$r \leq -1 \quad or \quad r \geq 1$	Sum$=\frac{a}{1-r}$
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Harmonic Series	$\sum_{n=1}^{\infty} \frac{1}{n}$	*No*	*Yes*	The harmonic series is a p-series with $p = 1$.
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$0 < a_{n+1} < a_n$ *and* $\lim_{n\to\infty} a_n = 0$		$\lvert\text{Error}\rvert \leq a_{n+1}$
Integral Test	$\sum_{n=1}^{\infty} a_n$ $a_n = f(n) \geq 0$	*if* $\int_1^{\infty} f(x)dx$ converges	*if* $\int_1^{\infty} f(x)dx$ diverges	$f(x)$ is continuous, positive and decreasing.
Limit Comparison $(a_n, b_n > 0)$	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ *and* $\sum_{n=1}^{\infty} b_n$ *converges*	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ *and* $\sum_{n=1}^{\infty} b_n$ *diverges*	Choose b_n that "behaves like" a_n.
Direct Comparison	$\sum_{n=1}^{\infty} a_n$	$0 < a_n < b_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converges*	$0 < b_n < a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *diverges*	
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	Test is inconclusive when $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} > 1$	Test is inconclusive when $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = 1$

Note: Telescoping Series Test questions can be tested using the Alternating Series Test conditions. On recent AP exams, the Telescoping Series Test is not listed as an answer choice, therefore it has been omitted in the chart. Since the harmonic series shows up frequently, it is recommended that students know this special case of the p-Series Test and that it is acceptable to cite the Harmonic Series Test as the justification for divergence when it occurs on the AP exam.

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Multiple Choice

1. (calculator not allowed)
   Which of the following series converge?

   I. $\sum_{n=1}^{\infty} \frac{n}{n+2}$

   II. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

   III. $\sum_{n=1}^{\infty} \frac{1}{n}$

   (A) None
   (B) II only
   (C) III only
   (D) I and II only
   (E) I and III only

2. (calculator not allowed)
   If $\lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$ is finite, then which of the following must be true?

   (A) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

   (B) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

   (C) $\sum_{n=1}^{\infty} \frac{1}{n^{p-2}}$ converges

   (D) $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ converges

   (E) $\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ diverges

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3. (calculator not allowed)

What is the value of $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}$?

(A) 1
(B) 2
(C) 4
(D) 6
(E) The series diverges

4. (calculator not allowed) (2003 BC22)

What are all values of p for which the infinite series $\sum_{n=1}^{\infty} \frac{n}{n^p + 1}$ converges?

(A) $p > 0$
(B) $p \geq 1$
(C) $p > 1$
(D) $p \geq 2$
(E) $p > 2$

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5. (calculator not allowed)

Which of the following series diverge?

I. $\sum_{n=0}^{\infty}\left(\frac{\sin 2}{\pi}\right)^n$

II. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$

III. $\sum_{n=1}^{\infty}\left(\frac{e^n}{e^n+1}\right)$

(A) III only
(B) I and II only
(C) I and III only
(D) II and III only
(E) I, II, and III

6. (calculator not allowed)

Consider the series $\sum_{n=1}^{\infty}\frac{e^n}{n!}$. If the ratio test is applied to the series, which of the following inequalities results, implying that the series converges?

(A) $\lim_{n\to\infty}\frac{e}{n!}<1$

(B) $\lim_{n\to\infty}\frac{n!}{e}<1$

(C) $\lim_{n\to\infty}\frac{n+1}{e}<1$

(D) $\lim_{n\to\infty}\frac{e}{n+1}<1$

(E) $\lim_{n\to\infty}\frac{e}{(n+1)!}<1$

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7. (calculator not allowed)
Which of the following series converges for all real numbers x?

(A) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

(B) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

(C) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

(D) $\sum_{n=1}^{\infty} \frac{e^n x^n}{n!}$

(E) $\sum_{n=1}^{\infty} \frac{n! x^n}{e^n}$

8. (calculator not allowed)

What are all values of x for which the series $\sum_{n=1}^{\infty} \left(\frac{2}{x^2+1} \right)^n$ converges?

(A) $-1 < x < 1$
(B) $x > 1$ only
(C) $x \geq 1$ only
(D) $x < -1$ and $x > 1$ only
(E) $x \leq -1$ and $x \geq 1$

9. (calculator allowed)

For what integer k, $k > 1$, will both $\sum_{n=1}^{\infty} \frac{(-1)^{kn}}{n}$ and $\sum_{n=1}^{\infty} \left(\frac{k}{4} \right)^n$ converge?

(A) 6
(B) 5
(C) 4
(D) 3
(E) 2

10. (calculator allowed)

What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$ converges?

(A) $-3 < x < -1$
(B) $-3 \leq x < -1$
(C) $-3 \leq x \leq -1$
(D) $-1 \leq x < 1$
(E) $-1 \leq x \leq 1$

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Free Response

11. (calculator not allowed)

A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \ldots + \frac{n+1}{3^{n+1}}x^n + \ldots$$

for all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\lim\limits_{x \to 0} \dfrac{f(x) - \frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

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12. (calculator not allowed)

The Maclaurin series for the function f is given by

$$f(x)=\sum_{n=0}^{\infty}\frac{(2x)^{n+1}}{n+1}=2x+\frac{4x^2}{2}+\frac{8x^3}{3}+\frac{16x^4}{4}+\ldots+\frac{(2x)^{n+1}}{n+1}+\ldots$$

on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series for *f*. Justify your answer.

(b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

(c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

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13. (calculator not allowed)

The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \ldots + \frac{(-1)^n nx^n}{n+1} + \ldots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots + \frac{(-1)^n x^n}{(2n)!} + \ldots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer

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14. (calculator not allowed)

Let f be the function given by $f(x) = \dfrac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\ln\left(1+x^2\right)$ is $\dfrac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1+x^2\right)$ about $x = 0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\dfrac{5}{4}\right)\right| < \dfrac{1}{100}$. Justify your answer.

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15. (calculator not allowed)

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty}\frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

(b Find the value of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$.

(c) Give a value of p such that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

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16. (calculator not allowed)

The Maclaurin series for the function f is given by $f(x)=\sum_{n=2}^{\infty}\frac{(-1)^n(2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

(b) Show that $y=f(x)$ is a solution to the differential equation $xy'-y=\frac{4x^2}{1+2x}$ for $|x|<R$, where R is the radius of convergence from part (a).

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